QUATERNIONIC REGULAR FUNCTIONS AND THE $\overline{\partial}$-NEUMANN PROBLEM IN $\mathbb{C}^2$

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1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where i, j, k denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$.

In this paper we give a boundary differential criterium that characterizes *(left) regular* functions $f : \Omega \to \mathbb{H}$ (in the sense of Fueter) among harmonic functions.

We show (Corollary 1) that there exist first order differential operators T and N, with complex coefficients, such that an harmonic function $f : \Omega \to \mathbb{H}$, of class C^1 on $\overline{\Omega}$, is regular if and only if $(N - jT)f = 0$ on $\partial\Omega$.

In order to obtain this result we study a related space of functions that satisfy a variant of the Cauchy-Riemann-Fueter equations, the space $M(\Omega)$ of *ψ-regular* functions on Ω (see §2 for the precise definitions) for the particular choice $\psi = \{1, i, j, -k\}$ of the structural vector. These functions (also called *ψ-hyperholomorphic functions*) have been studied by many authors (see for instance [SV], [MS] and [No]). The space $M(\Omega)$ contains the holomorphic maps and exhibits interesting links with the theory of two complex variables. In particular, Vasilevski and Shapiro [VS] have shown that the Bochner-Martinelli kernel $U(\zeta, z)$ can be considered as a first complex component of the Cauchy-Fueter kernel associated to ψ-regular functions. This property had already been observed by Fueter (see [F]) in the general n-dimensional case, by means of an imbedding of $\mathbb{C}^n$ in a real Clifford algebra. Note that regular functions are in a simple correpondence with ψ-regular functions, since they can be obtained from them by means of a real coordinate reflection in $\mathbb{H}$.

We prove (Theorem 1) that an harmonic function f on Ω, of class C^1 on $\overline{\Omega}$, is ψ-regular on Ω if and only if $(\overline{\partial}_n - jL)f = 0$ on $\partial\Omega$, where $\overline{\partial}_n$ is the normal part of $\overline{\partial}$ and L is a tangential Cauchy-Riemann operator.

This equation, which appeared in [P] in connection with the characterization of the traces of pluriharmonic functions, can be considered as a generalization both of the CR-tangential equation $L(f) = 0$ and of the condition $\overline{\partial}_n f = 0$ on $\partial\Omega$ that distinguishes holomorphic functions among complex harmonic functions (Aronov and Kytmanov (see [A], [AK] and [KA]).

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As an application of the differential condition for ψ-regular functions, we also obtain (Theorem 4) a differential criterium for holomorphicity of functions that generalizes, for a domain with connected boundary in $\mathbb{C}^2$, the result of Aronov and Kytmanov.

In Theorem 2 and Theorem 3 we give weak versions of the results stated above and obtain trace theorems for ψ-regular functions, with applications to regular and holomorphic functions. Other results on this line have been given by Pertici in [Pe], with generalizations to several quaternionic variables.

In §4 we study the relation between regularity and the $\overline{\partial}$-Neumann problem in $\mathbb{C}^2$ in the formulation given in [K]§14–18.

In particular we show (Corollary 5) that if Ω is a strongly pseudoconvex domain of class C^∞ or a weakly pseudoconvex domain with real-analytic boundary, then the operator that associates to $f = f_1 + f_2 j$ the restriction to $\partial\Omega$ of its first complex component f_1 induces an isomorphism between the quotient spaces $M^\infty(\Omega)/A^\infty(\Omega, \mathbb{C}^2)$ and $C^\infty(\partial\Omega)/CR(\partial\Omega)$, where $M^\infty(\Omega)$ denotes the space of ψ-regular functions that are smooth up to the boundary and $A^\infty(\Omega, \mathbb{C}^2)$ is the space $Hol(\Omega, \mathbb{C}^2) \cap C^\infty(\overline{\Omega}, \mathbb{C}^2)$

Finally, in §5 we consider the particular case when Ω is the unit ball B in $\mathbb{C}^2$ and $S = \partial B$ is the group of unit quaternions. We study in more detail the regular homogeneous polynomials, which appear as components in the power series of any regular function. In Theorem 6 and Corollary 6 we obtain differential conditions that characterize the homogeneous polynomials whose restrictions to S extend regularly into B. This result generalizes the analogous characterization for holomorphic extensions of polynomials proved by Kytmanov in [K1]. In §5.2 we make more explicit the isomorphism of Corollary 5 by means of (complex) spherical harmonics. In particular, we show (Theorem 8) how to construct bases of the right $\mathbb{H}$-module of (left)-regular homogeneous polynomials of a fixed degree starting from any choice of $\mathbb{C}$-bases of the spaces of complex harmonic homogeneous polynomials.

Some of the results contained in the present paper have been announced in [P1].

2. Notations and preliminaries

2.1. Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^n$ with boundary of class $C^m, m \geq 1$. We assume $\rho \in C^m$ on $\mathbb{C}^n$ and $d\rho \neq 0$ on $\partial\Omega$.

Let ν denote the outer unit normal to $\partial\Omega$ and $\tau = i\nu$. For every $f \in C^1(\overline{\Omega})$, we set $\overline{\partial}_n f = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial \nu} + i\dfrac{\partial f}{\partial \tau}\right)$ (see [K]§§3.3 and 14.2).

Then in a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} f$ in the tangential and the normal parts

$$\overline{\partial} f = \overline{\partial}_b f + \overline{\partial}_n f \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}.$$

The normal part of $\overline{\partial} f$ on $\partial\Omega$ can also be expressed in the form

$$\overline{\partial}_n f = \sum_k \frac{\partial f}{\partial \bar{z}_k} \frac{\partial \rho}{\partial z_k} \frac{1}{|\overline{\partial}\rho|},$$

where $|\overline{\partial}\rho|^2 = \sum_{k=1}^n \left|\dfrac{\partial\rho}{\partial\bar{z}_k}\right|^2$, or, by means of the Hodge $*$-operator and the Lebesgue surface measure $d\sigma$, as $\overline{\partial}_n f d\sigma = *\overline{\partial} f_{|\partial\Omega}$.

2.2. We recall the definition of tangential Cauchy-Riemann operators (see for example [R]§18). A linear first-order differential operator L is *tangential* to $\partial\Omega$ if $(L\rho)(\zeta) = 0$ for each point $\zeta \in \partial\Omega$. A tangential operator of the form

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}$$

is called a *tangential Cauchy-Riemann* operator. The operators

$$L_{jk} = \frac{1}{|\bar{\partial}\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_k}\frac{\partial}{\partial\bar{z}_j} - \frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial}{\partial\bar{z}_k}\right), \quad 1 \le j < k \le n,$$

are tangential and the corresponding vectors at $\zeta \in \partial\Omega$ span (not independently when $n > 2$) the (conjugate) complex tangent space to $\partial\Omega$ at ζ. Then a function $f \in C^1(\partial\Omega)$ is a CR function if and only if $L_{jk}(f) = 0$ on $\partial\Omega$ for every j, k, or, equivalently, if $L(f) = 0$ for each tangential Cauchy-Riemann operator L. In particular, when $n = 2$, f is a CR function if and only if $L(f) = 0$ on $\partial\Omega$.

2.3. Let $\mathbb{H}$ be the algebra of quaternions. The elements of $\mathbb{H}$ have the form

$$q = x_0 + ix_1 + jx_2 + kx_3,$$

where x_0, x_1, x_2, x_3 are real numbers and i, j, k denote the basic quaternions.

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j$. The commutation rule is then $aj = j\bar{a}$ for every $a \in \mathbb{C}$, and the quaternionic conjugation is

$$\bar{q} = x_0 - ix_1 - jx_2 - kx_3 = \bar{z}_1 - z_2 j.$$

We refer to [S] for the basic facts of quaternionic analysis. We will denote by $\mathcal{D}$ the left Cauchy-Riemann-Fueter operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is *(left-)regular* on a domain Ω, that is $\mathcal{D}f = 0$, if and only if the complex components (f_1, f_2) of f satisfy the following system of complex differential equations on Ω:

$$(1) \qquad \begin{cases} \dfrac{\partial f_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{f_2}}{\partial \bar{z}_2} \\[2ex] \dfrac{\partial f_1}{\partial z_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1} \end{cases}$$

We will use also another class of regular functions, which are in the kernel of the following left differential operator:

$$\mathcal{D}' = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right).$$

This is the Cauchy-Riemann operator defined by the structural vector

$$\psi = \{1, i, j, -k\}.$$

A quaternionic C^1 function $f = f_1 + f_2 j$, is called *(left-)ψ-regular* on a domain Ω, if $\mathcal{D}' f = 0$ on Ω. This condition is equivalent to the following system of complex differential equations:

$$\left\{ \begin{array}{l} \dfrac{\partial f_1}{\partial \bar{z}_1} = \dfrac{\partial \overline{f_2}}{\partial z_2} \\ \dfrac{\partial f_1}{\partial \bar{z}_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1} \end{array} \right. \tag{2}$$

or to the equation $*\overline{\partial} f_1 = -\frac{1}{2}\partial(\overline{f_2} d\bar{z}_1 \wedge d\bar{z}_2)$. Note that the $(1,2)$-form $*\overline{\partial} f_1$ is ∂-closed on Ω when f_1 is harmonic. Then, if Ω is a pseudoconvex domain in $\mathbb{C}^2$, every complex harmonic function f_1 on Ω is a complex component of a ψ-regular function $f = f_1 + f_2 j$. (cf. [N] and [No1] for this result and its converse).

Remark. Any holomorphic mapping (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$. Moreover, the complex components of a ψ-regular function are either both holomorphic or both not-holomorphic.

We refer, for instance, to [SV], [MS] and [No] for the properties of structural vectors and ψ-regular functions (in these references they are called ψ-hyperholomorphic functions). Regular and ψ-regular functions are real analytic on Ω, and they are harmonic with respect to the Laplace operator in $\mathbb{R}^4$.

Remark. Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if, and only if, $f \circ \gamma$ is ψ-regular on $\gamma^{-1}(\Omega)$.

2.4. Let's denote by G the Cauchy-Fueter quaternionic kernel defined by

$$G(p - q) = \frac{1}{2\pi^2} \frac{\bar{p} - \bar{q}}{|p - q|^4},$$

and by G' the Cauchy kernel for ψ-regular functions:

$$G'(p - q) = \frac{1}{2\pi^2} \frac{y_0 - x_0 - i(y_1 - x_1) - j(y_2 - x_2) + k(y_3 - x_3)}{|p - q|^4},$$

where $p = y_0 + iy_1 + jy_2 + ky_3$, $q = x_0 + ix_1 + jx_2 + kx_3$.

Let $\sigma(q)$ be the quaternionic valued 3-form

$$\sigma(q) = dx[0] - i dx[1] + j dx[2] + k dx[3],$$

where $dx[k]$ denotes the product of dx_0, dx_1, dx_2, dx_3 with dx_k deleted. Then the Cauchy-Fueter integral formula for left-ψ-regular functions on Ω that are continuous on $\overline{\Omega}$, holds true:

$$\int_{\partial\Omega} G'(p - q)\sigma(p) f(p) = \begin{cases} f(q) & \text{for } q \in \Omega, \\ 0 & \text{for } q \notin \overline{\Omega}. \end{cases}$$

In [VS] (see also [F] and [MS]) it was shown that, for a family of structural vectors, including $\{1, i, j, -k\}$, the two-dimensional Bochner-Martinelli form $U(\zeta, z)$ can be considered as a first complex component of the Cauchy-Fueter kernel associated to ψ-regular functions. Let $q = z_1 + z_2 j$, $p = \zeta_1 + \zeta_2 j$. Then

$$G'(p-q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j,$$

where $\omega(\zeta, z)$ is the following complex $(1,2)$-form:

$$\omega(\zeta, z) = -\frac{1}{4\pi^2|\zeta - z|^4}\left((\bar{\zeta}_1 - \bar{z}_1)d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2)d\zeta_2\right) \wedge \overline{d\zeta}.$$

Here $d\zeta = d\zeta_1 \wedge d\zeta_2$ and we choose the orientation of $\mathbb{C}^2$ given by the volume form $\frac{1}{4}dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$.

3. Differential criteria for regularity and holomorphicity

3.1. We now rewrite the representation formula of Cauchy-Fueter for ψ-regular functions in complex form. We use results from [MS] to relate the form $\omega(\zeta, z)$ to the tangential operator L_{12}, that we will denote simply by L. We show that the Bochner-Martinelli formula can then be applied to obtain a criterium that distinguishes regular functions among harmonic functions on a domain Ω in $\mathbb{C}^2 = \mathbb{H}$.

Let $g(\zeta, z) = \frac{1}{4\pi^2}|\zeta - z|^{-2}$ be the fundamental solution of the complex laplacian on $\mathbb{C}^2$.

Proposition 1. *Let Ω be a bounded domain of class C^1 in $\mathbb{H}$. Let $f : \Omega \to \mathbb{H}$ be a quaternionic function, of class C^1 on $\overline{\Omega}$. Then f is (left-)ψ-regular on Ω if, and only if, the following representation formula holds on Ω:*

$$f(z) = \int_{\partial\Omega} U(\zeta, z)f(\zeta) + 2\int_{\partial\Omega} g(\zeta, z)jL(f(\zeta))d\sigma$$

where $d\sigma$ is the Lebesgue measure on $\partial\Omega$ and the tangential operator L acts on $f = f_1 + f_2 j$ as $L(f) = L(f_1) + L(f_2)j$.

Proof. The integral of Cauchy-Fueter in complex form is

$$\int_{\partial\Omega} G'(p-q)\sigma(p)f(p) = \int_{\partial\Omega} U(\zeta, z)f(\zeta) + \int_{\partial\Omega} \omega(\zeta, z)jf(\zeta).$$

From Proposition 6.3 in [MS], we get that the last integral is equal to

$$\begin{aligned}\int_{\partial\Omega} \omega(\zeta, z)\overline{f_1}j - \int_{\partial\Omega} \omega(\zeta, z)\overline{f_2} &= 2\int_{\partial\Omega} g(\zeta, z)\left(\overline{L(f_1)}j - \overline{L(f_2)}\right)d\sigma \\ &= 2\int_{\partial\Omega} g(\zeta, z)jL(f)d\sigma.\end{aligned}$$

Then the result follows from the Cauchy-Fueter integral formula for ψ-regular functions.

If $f = f_1 + f_2 j$ is a ψ-regular function on Ω, of class C^1 on $\overline{\Omega}$, then from equations (2) we get that it satisfies the equation

$$(\overline{\partial}_n - jL)f = 0 \quad \text{on } \partial\Omega, \tag{3}$$

since $\overline{\partial}_n f_1 = -\overline{L(f_2)}$, $\overline{\partial}_n f_2 = \overline{L(f_1)}$ on $\partial\Omega$.

This equation was introduced in [P]§4 in connection with the characterization of the traces of pluriharmonic functions. It can be considered as a generalization both of the CR-tangential equation $L(f) = 0$ (for a complex-valued f) and of the condition $\overline{\partial}_n f = 0$ on $\partial\Omega$ that distinguishes holomorphic functions among complex harmonic functions on Ω (what is called the homogeneous $\overline{\partial}$-Neumann problem for functions, see [AK] and [K]§15).

Theorem 1. *Let Ω be a bounded domain in $\mathbb{H}$, with boundary of class C^1. Let $f = f_1 + f_2 j : \Omega \to \mathbb{H}$ be an harmonic function on Ω, of class C^1 on $\overline{\Omega}$. Then, f is (left-)ψ-regular on Ω if, and only if,*

$$(\overline{\partial}_n - jL)f = 0 \quad \text{on } \partial\Omega.$$

Proof. It remains to prove the sufficiency of condition (3) for ψ-regularity of harmonic functions. For every $z \in \Omega$, it follows from the Bochner-Martinelli integral representation for complex harmonic functions on Ω (see for example [K]§1.1), that $f(z) = f_1(z) + f_2(z)j$ is equal to

$$\int_{\partial\Omega} U(\zeta, z) f_1(\zeta) + 2\int_{\partial\Omega} g(\zeta, z)\overline{\partial}_n f_1(\zeta) d\sigma \\ + \left(\int_{\partial\Omega} U(\zeta, z) f_2(\zeta) + 2\int_{\partial\Omega} g(\zeta, z)\overline{\partial}_n f_2(\zeta) d\sigma\right) j.$$

If $\overline{\partial}_n f = jL(f)$ on $\partial\Omega$, then we obtain

$$f(z) = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + 2\int_{\partial\Omega} g(\zeta, z)\overline{\partial}_n f(\zeta) d\sigma \\ = \int_{\partial\Omega} U(\zeta, z) f(\zeta) + 2\int_{\partial\Omega} g(\zeta, z) jL(f(\zeta)) d\sigma.$$

The result now follows from Proposition 1.

Let N and T be the differential operators, defined in a neighbourhood of $\partial\Omega$,

$$N = \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial\bar{z}_1} + \frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial z_2}, \quad T = \frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial z_2}.$$

T is a tangential (not Cauchy-Riemann) operator to $\partial\Omega$, while N is non-tangential, such that $N(\rho) = |\overline{\partial}\rho|^2$, $\mathrm{Re}(N) = |\overline{\partial}\rho|\,\mathrm{Re}(\overline{\partial}_n)$. The remark made at the end of §2.3 shows that Theorem 1 gives also a boundary condition for regularity of an harmonic function on Ω.

Corollary 1. *Let Ω be a C^1-bounded domain in $\mathbb{H}$. Let $f = f_1 + f_2 j : \Omega \to \mathbb{H}$ be an harmonic function on Ω, of class C^1 on $\overline{\Omega}$. Then, f is (left-)regular on Ω if, and only if,*

$$(N - jT)f = 0 \quad \text{on } \partial\Omega.$$

3.2. We now give a weak formulation of the differential criterium of ψ-regularity, which makes sense for example when the harmonic function f is only continuous on the closure $\overline{\Omega}$.

Let $Harm^1(\overline{\Omega})$ denote the space of complex harmonic functions on Ω, of class C^1 on $\overline{\Omega}$. By application of the Stokes' Theorem, of the complex Green formula

$$\int_{\partial\Omega} g\overline{\partial}_n h d\sigma = \int_{\partial\Omega} h\partial_n g d\sigma \quad \forall\, g, h \in Harm^1(\overline{\Omega}),$$

and of the equality $\overline{\partial} f \wedge d\zeta_{|\partial\Omega} = 2L(f)d\sigma$ on $\partial\Omega$, we see that the equations $\overline{\partial}_n f_1 = -\overline{L(f_2)}$, $\overline{\partial}_n f_2 = \overline{L(f_1)}$ on $\partial\Omega$, imply the following (complex) integral conditions: for every function $\phi \in Harm^1(\overline{\Omega})$,

$$\int_{\partial\Omega} \overline{f_1} {*}\overline{\partial}\phi = \frac{1}{2}\int_{\partial\Omega} f_2\overline{\partial}(\phi d\zeta), \quad \int_{\partial\Omega} \overline{f_2} {*}\overline{\partial}\phi = -\frac{1}{2}\int_{\partial\Omega} f_1\overline{\partial}(\phi d\zeta). \tag{4}$$

These are equivalent to one quaternionic condition, which is then necessary for the ψ-regularity of $f \in C^1(\overline{\Omega})$:

$$\int_{\partial\Omega} \bar{f}\left({*}\overline{\partial}\phi - \frac{1}{2}j\overline{\partial}(\phi d\zeta)\right) = 0 \quad \forall\, \phi \in Harm^1(\overline{\Omega}),$$

that can be rewritten also as:

$$\int_{\partial\Omega} \bar{f}\left(\overline{\partial}_n - jL\right)(\phi)\, d\sigma = 0 \quad \forall\, \phi \in Harm^1(\overline{\Omega}). \tag{5}$$

Now we consider the sufficiency of the integral condition (5) when f is only continuous on $\overline{\Omega}$.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{H}$, with boundary of class C^1. Let $f : \partial\Omega \to \mathbb{H}$ be a continuous function. Then, there exists a (left-)ψ-regular function F on Ω, continuous on $\overline{\Omega}$, such that $F_{|\partial\Omega} = f$, if and only if f satisfies the condition (5).*

Proof. Let F^+ and F^- be the ψ-regular functions defined respectively on Ω and on $\mathbb{C}^2 \setminus \overline{\Omega}$ by the Cauchy-Fueter integral of f:

$$F^{\pm}(z) = \int_{\partial\Omega} U(\zeta, z)f(\zeta) + \int_{\partial\Omega} \omega(\zeta, z)jf(\zeta).$$

From the equalities $U(\zeta, z) = -2{*}\partial_\zeta g(\zeta, z)$, $\omega(\zeta, z) = -\partial_\zeta(g(\zeta, z)d\bar{\zeta})$, we get that

$$\overline{F^-(z)} = -2\int_{\partial\Omega} \overline{f(\zeta)}{*}\overline{\partial}_\zeta g(\zeta, z) + \int_{\partial\Omega} \overline{f(\zeta)}j\overline{\partial}_\zeta(g(\zeta, z)d\zeta)$$

for every $z \notin \overline{\Omega}$. If (5) holds, then F^- vanishes identically on $\mathbb{C}^2 \setminus \Omega$. From the Sokhotski-Plemelj formula (see [VS]§3.6), we then obtain that F^+ is continuous on $\overline{\Omega}$, and $F^+ = f$ on $\partial\Omega$. Conversely, if $F \in C(\overline{\Omega})$ is a ψ-regular function on Ω with trace f on $\partial\Omega$, and $\Omega_\epsilon = \{z \in \Omega : \rho < \epsilon\}$, then F satisfies (5) on $\partial\Omega_\epsilon$ for every small $\epsilon < 0$. Passing to the limit as $\epsilon \to 0$, we obtain (5).

Remark. Using the results of §3.7 in [VS], the same conclusion can be achieved for functions f in $L^p(\partial\Omega)$. Moreover, in the orthogonality condition (5) it is sufficient to consider functions $\phi \in Harm^1(\overline{\Omega})$ that are of class C^∞ on a neighbourhood of $\overline{\Omega}$.

3.3. In Theorem 2, the boundary of Ω is not required to be connected. If $\partial\Omega$ is connected, we can improve the result and show that only one of the complex conditions (4) is sufficient for the ψ-regularity of the harmonic extension of f.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{H}$, with connected boundary $\partial\Omega$ of class C^1. Let $f : \partial\Omega \to \mathbb{H}$ be a continuous function. Then, if f satisfies one of the conditions (4), there exists a (left-)ψ-regular function F on Ω, continuous on $\overline{\Omega}$, such that $F_{|\partial\Omega} = f$.*

Proof. Assume that

$$\int_{\partial\Omega} \overline{f_2} {*}\overline{\partial}\phi = -\frac{1}{2}\int_{\partial\Omega} f_1\overline{\partial}(\phi d\zeta) \quad \forall\ \phi \in Harm^1(\overline{\Omega}).$$

We use the same notation as in the proof of Theorem 2. We get that

$$\begin{aligned}\overline{F^-(z)} = -2\int_{\partial\Omega} \overline{f(\zeta)}{*}\overline{\partial}_\zeta g(\zeta,z) + \int_{\partial\Omega} \overline{f(\zeta)}j\overline{\partial}_\zeta(g(\zeta,z)d\zeta) \\ = -2\int_{\partial\Omega} \overline{f_1(\zeta)}{*}\overline{\partial}_\zeta g(\zeta,z) + \int_{\partial\Omega} f_2(\zeta)\overline{\partial}_\zeta(g(\zeta,z))\end{aligned}$$

for every $z \notin \overline{\Omega}$. Therefore, F^- is a complex-valued, ψ-regular function on $\mathbb{C}^2 \setminus \overline{\Omega}$. The system of equations (2) then implies that F^- is an holomorphic function. Since $\partial\Omega$ is connected, from Hartogs' Theorem it follows that F^- can be holomorphically continued to the whole space. Let $\tilde{F}^-$ be such extension. Then $F = F^+ - \tilde{F}^-_{|\Omega}$ is a ψ-regular function on Ω, continuous on $\overline{\Omega}$, such that $F_{|\partial\Omega} = f$. If the first condition in (4) is satisfied, it is sufficient to consider the function $fj = -f_2 + f_1 j$ in place of f.

Remarks. (1) The hypothesis on f in the preceding theorem is satisfied, for example, when f is of class C^1 on $\overline{\Omega}$, harmonic on Ω, and one of the equations $\overline{\partial}_n f_1 = -\overline{L(f_2)}$, $\overline{\partial}_n f_2 = \overline{L(f_1)}$ holds on $\partial\Omega$.

(2) The connectedness of $\partial\Omega$ is a necessary condition in Theorem 3: consider a locally constant function on $\partial\Omega$.

The preceding result can be easily generalized in the following form:

Corollary 2. *Let Ω be as above. Let $a, b \in \mathbb{C}$ be two complex numbers such that $(a, b) \neq (0, 0)$.*

(i) If f is of class C^1 on $\overline{\Omega}$, harmonic on Ω, and such that

$$a\ \overline{\partial}_n f_1 + b\ \overline{\partial}_n f_2 = -a\ \overline{L(f_2)} + b\ \overline{L(f_1)} \quad \text{on } \partial\Omega,$$

then f is ψ-regular on Ω.

(ii) If f is continuous on $\partial\Omega$, such that

$$\int_{\partial\Omega} (a\overline{f_1} + b\overline{f_2}){*}\overline{\partial}\phi = \frac{1}{2}\int_{\partial\Omega} (af_2 - bf_1)\overline{\partial}(\phi d\zeta) \quad \forall\ \phi \in Harm^1(\overline{\Omega}),$$

then there exists a ψ-regular function F on Ω, continuous on $\overline{\Omega}$, such that $F_{|\partial\Omega} = f$.

As we did in Corollary 1, from Theorem 3 we can deduce a boundary condition for regularity of an harmonic function on Ω, when $\partial\Omega$ is connected.

Corollary 3. *Let Ω be as above. Let $f = f_1 + f_2 j : \Omega \to \mathbb{H}$ be an harmonic function on Ω, of class C^1 on $\overline{\Omega}$. Let N and T be the differential operators introduced in §3.1. Then, the following conditions are equivalent:*

(i) f is (left-)regular on Ω.

(ii) $N(f_1) = -\overline{T(f_2)}$ on $\partial\Omega$.

(iii) $N(f_2) = \overline{T(f_1)}$ on $\partial\Omega$.

(iv) $aN(f_1) + bN(f_2) = -a\overline{T(f_2)} + b\overline{T(f_1)}$ on $\partial\Omega$, for some $a, b \in \mathbb{C}$, $(a, b) \neq (0, 0)$.

Theorem 3 can be applied, in the case of connected boundary in $\mathbb{C}^2$, to obtain the following result of Aronov and Kytmanov (cf. [A] and [AK]), which holds in $\mathbb{C}^n$, $n > 1$: if f is a complex harmonic function on Ω, of class C^1 on $\overline{\Omega}$, such that $\overline{\partial}_n f = 0$ on $\partial\Omega$, then f is holomorphic. It is sufficient to take $f_1 = f, f_2 = 0$.

More generally, we can deduce a differential criterium for holomorphicity of functions on a domain with connected boundary in $\mathbb{C}^2$, analogous to those proposed in [C] and investigated in [K1] and [K]§23.2.

Theorem 4. *Let Ω be a bounded domain in $\mathbb{C}^2$, with connected boundary of class C^1. Let $h = (h_1, h_2) : \Omega \to \mathbb{C}^2$ be an holomorphic mapping of class C^1 on $\overline{\Omega}$, such that $h(\zeta) \neq 0$ for every $\zeta \in \partial\Omega$.*

(i) If $h_1 f, h_2 f \in Harm^1(\overline{\Omega})$ and $f : \overline{\Omega} \to \mathbb{C}$ satisfies the differential condition

$$h_1 \overline{\partial}_n f = \overline{h_2 L(f)} \quad \text{on } \partial\Omega,$$

then f is holomorphic on Ω.

(ii) If $h_1 f$ and $h_2 f$ are harmonic on Ω, $f : \overline{\Omega} \to \mathbb{C}$ is continuous and it satisfies the integral condition

$$\int_{\partial\Omega} \overline{h_1 f} * \overline{\partial}\phi = \int_{\partial\Omega} h_2 f \overline{\partial}(\phi d\zeta) \quad \forall\ \phi \in Harm^1(\overline{\Omega}),$$

then f is holomorphic on Ω.

Proof. It is sufficient to prove (ii). Let $h_2' = -2h_2$. From Theorem 3 we get that the harmonic function $h_2' f + h_1 f j$ is ψ-regular on Ω. Let $\Omega_\epsilon = \{z \in \Omega : \rho < \epsilon\}$ for a small $\epsilon < 0$. Then the following equalities hold on $\partial\Omega_\epsilon$:

$$\overline{\partial}_n(h_1 f) = \overline{L(h_2' f)}, \quad \overline{\partial}_n(h_2' f) = -\overline{L(h_1 f)}.$$

From the holomorphicity of h, we then obtain

$$h_1 \overline{\partial}_n(f) = \bar{h}_2' \overline{L(f)}, \quad h_2' \overline{\partial}_n(f) = -\bar{h}_1 \overline{L(f)} \quad \text{on } \partial\Omega_\epsilon,$$

which implies $\overline{\partial}_n f = L(f) = 0$ on $\partial\Omega_\epsilon$ for every ϵ sufficiently small, such that $h \neq 0$ on $\partial\Omega_\epsilon$. This means that there exists an holomorphic extension F_ϵ of f on Ω_ϵ. From the equality of the harmonic functions $h_j F_\epsilon = h_j f$ on Ω_ϵ, for $j = 1, 2$, we get $F_\epsilon = f_{|\Omega_\epsilon}$. Then f is holomorphic on the whole domain Ω.

Remark. The class of functions $f(z_1, z_2)$, such that $h_1 f, h_2 f$ are harmonic, h_1, h_2 holomorphic, is wide also in the case when h_1, h_2 are non-constant. For example, if $h_1 = z_1, h_2 = 1$ and f is any harmonic function, holomorphic with respect to z_1, then $\Delta(h_1 f) = \Delta(h_2 f) = 0$.

4. Regularity and the $\overline{\partial}$-Neumann problem

4.1. The $\overline{\partial}$-Neumann problem for complex functions $\Box f = \psi$ in Ω, $\overline{\partial}_n f = 0$ on $\partial\Omega$ is equivalent, in the smooth case, to the problem

$$\overline{\partial}_n g = \phi \text{ on } \partial\Omega, \quad g \text{ harmonic in } \Omega$$

(see [K]§14). The compatibility condition for this problem is

$$\int_{\partial\Omega} \phi \overline{h} d\sigma = 0 \tag{6}$$

for every h holomorphic in a neighbourhood of $\overline{\Omega}$. We now use the solvability of this problem in strongly pseudoconvex domains of $\mathbb{C}^2$ to obtain some results on regular functions.

Let Ω be a bounded domain in $\mathbb{C}^2$ with connected, C^∞-smooth boundary. We denote by $W^s(\Omega)$ $(s \geq 1)$ the complex Sobolev space, and by $G^s(\Omega)$ the space of harmonic functions in $W^s(\Omega)$. The space $G^s(\Omega)$ is isomorphic to $W^{s-1/2}(\partial\Omega)$ through the restriction operator.

Theorem 5. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^2$ with connected boundary of class C^∞. Let $f_1 \in W^{s-1/2}(\partial\Omega)$, where $s \geq 3$. We identify f_1 with its harmonic extension in $G^s(\Omega)$. Then there exists a function $f_2 \in G^{s-2}(\Omega)$ such that $f = f_1 + f_2 j$ is a ψ-regular function on Ω.*

Proof. We show that the function $\phi = \overline{L(f_1)} \in G^{s-1}(\Omega)$ satisfies the compatibility condition (6). If h is holomorphic in a neighbourhood of $\overline{\Omega}$,

$$\int_{\partial\Omega} L(f_1) h d\sigma = \frac{1}{2}\int_{\partial\Omega} h\overline{\partial}(f_1 dz) = 0,$$

since h is a CR function on $\partial\Omega$. Then we can apply a result of Kytmanov ([K]§18.2) and get a solution $f_2 \in G^{s-2}(\Omega)$ of the $\overline{\partial}$-Neumann problem $\overline{\partial}_n f_2 = \overline{L(f_1)}$ on $\partial\Omega$. If $s \geq 5$, then $f = f_1 + f_2 j$ is continuous on $\overline{\Omega}$ by Sobolev embedding. From Theorem 3 we get that f is ψ-regular on Ω, since it satisfies the second condition in (4). In any case, $f_2 \in L^2(\partial\Omega)$ since $s \geq 3$. Then the result follows from the $L^2(\partial\Omega)$-version of Theorem 3, that can be proved as before using the results in [VS]§3.7.

Remark. There is a unique solution f_2 of $\overline{\partial}_n f_2 = \overline{L(f_1)}$ on $\partial\Omega$ that is orthogonal to holomorphic functions in $L^2(\partial\Omega)$. It is given by the bounded Neumann operator N_Ω: $f_2 = N_\Omega(\overline{L(f_1)})$.

Corollary 4. *Suppose Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^2$ with connected boundary of class C^∞. Let $f_1 : \partial\Omega \to \mathbb{C}$ be of class C^∞. Then there exists a ψ-regular function f on Ω, of class C^∞ on $\overline{\Omega}$, such that the first complex component of the restriction $f_{|\partial\Omega}$ to $\partial\Omega$ is f_1.*

Remark. The preceding statements remain true if Ω is a bounded weakly pseudoconvex domain in $\mathbb{C}^2$ with connected real-analytic boundary, since on these domains the $\overline{\partial}$-Neumann problem for smooth functions is solvable (cf. [K]§18).

4.2. We denote by $M(\Omega)$ the right $\mathbb{H}$-module of (left-)ψ-regular functions on Ω and by $M^\infty(\Omega)$ the functions in $M(\Omega)$ that are of class C^∞ on $\overline{\Omega}$. We consider the space $Hol(\Omega,\mathbb{C}^2)$ as a real subspace of $M(\Omega)$ by identification of the map (f_1,f_2) with $f=f_1+f_2j$.

If Ω is pseudoconvex, it follows from what observed in §2.3 that the map that associates to $f=f_1+f_2j$ the first complex component f_1 induces an isomorphism between the quotient real spaces $M(\Omega)/Hol(\Omega,\mathbb{C}^2)$ and $Harm(\Omega)/\mathcal{O}(\Omega)$.

Now we are also interested in the regularity up to the boundary of the functions. Let $A^\infty(\Omega,\mathbb{C}^2)=Hol(\Omega,\mathbb{C}^2)\cap C^\infty(\overline{\Omega},\mathbb{C}^2)$ be identified with a $\mathbb{R}$-subspace of $M^\infty(\Omega)$.

Let $C: M^\infty(\Omega)\to C^\infty(\partial\Omega)$ be the linear operator that associates to $f=f_1+f_2j$ the restriction to $\partial\Omega$ of its first complex component f_1. From the Corollary 4 and the remark preceding it, we get a right inverse R of C. The function $R(f_1)$ is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of $\overline{\Omega}$:

$$\int_{\partial\Omega}(R(f_1)-f_1)\overline{h}d\sigma=0 \quad \forall h\in\mathcal{O}(\overline{\Omega}).$$

Note that $f_1\in CR(\partial\Omega)\cap C^\infty(\partial\Omega)$ if and only if $R(f_1)\in A^\infty(\Omega,\mathbb{C}^2)$. Besides, the operator C has kernel

$$\ker C=\{f_2j : f_2\in A^\infty(\Omega)\}=A^\infty(\Omega)j$$

where $A^\infty(\Omega)$ is the space of the holomorphic functions on Ω that are C^∞ up to the boundary. Then C induces the following isomorphism of real spaces:

$$\tilde{C} \ : \ \frac{M^\infty(\Omega)}{A^\infty(\Omega)j}\to C^\infty(\partial\Omega).$$

Corollary 5. *Let Ω be a bounded domain in $\mathbb{C}^2$ with connected boundary. Suppose that Ω is a strongly pseudoconvex domain of class C^∞ or a weakly pseudoconvex domain with real-analytic boundary. Then the operator C induces an isomorphism of real spaces:*

$$\hat{C} \ : \ \frac{M^\infty(\Omega)}{A^\infty(\Omega,\mathbb{C}^2)}\to\frac{C^\infty(\partial\Omega)}{CR(\partial\Omega)}.$$

Remark. If Ω is a C^∞-smooth bounded pseuoconvex domain, from application of Kohn's Theorem on the solvability of the $\overline{\partial}$-problem to the equation $*\overline{\partial}f_1=-\frac{1}{2}\partial(\overline{f_2}d\bar{z}_1\wedge d\bar{z}_2)$ we can still deduce the isomorphism of Corollary 5.

5. The case of the unit ball in $\mathbb{C}^2$

5.1. Let Ω be the unit ball B in $\mathbb{C}^2$, $S=\partial B$ the unit sphere. In this case the operators $\overline{\partial}_n$ and L have the following forms:

$$\overline{\partial}_n=\bar{z}_1\frac{\partial}{\partial\bar{z}_1}+\bar{z}_2\frac{\partial}{\partial\bar{z}_2},\qquad L=z_2\frac{\partial}{\partial\bar{z}_1}-z_1\frac{\partial}{\partial\bar{z}_2}$$

and they preserve harmonicity. Theorem 3 and Corollary 2 can be applied to get differential conditions that characterize the homogeneous polynomials whose restrictions to S extend regularly (or ψ-regularly) into B. We use a computation made by Kytmanov in [K1] (cf. also [K] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$ be the Laplacian in $\mathbb{C}^2$ and D_k the differential operator

$$D_k = \sum_{0 \le l \le k/2-1} \frac{(k-2l-1)!(2l-1)!!}{k!(l+1)!} 2^l \Delta^{l+1}.$$

Theorem 6. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a ψ-regular function into B if, and only if,*

$$(\overline{\partial}_n - D_k) f_1 + \overline{L(f_2)} = 0 \quad \text{on } S.$$

More generally, the harmonic extension of $f_{|S}$ is ψ-regular on B if there exists a pair of complex numbers $(a,b) \neq (0,0)$ such that

$$(\overline{\partial}_n - D_k)(a f_1 + b f_2) + \overline{L(\bar{a} f_2 - \bar{b} f_1)} = 0 \quad \text{on } S.$$

Proof. We can proceed as in [K1]. The harmonic extension $\tilde{f}_1$ of $f_{1|S}$ into B is given by Gauss's formula: $\tilde{f}_1 = \sum_{s \ge 0} g_{k-2s}$, where g_{k-2s} is the homogeneous harmonic polynomial of degree $k-2s$ defined by

$$g_{k-2s} = \frac{k-2s+1}{s!(k-s+1)!} \sum_{j \ge 0} \frac{(-1)^j (k-j-2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1. \tag{6}$$

Then $\overline{\partial}_n \tilde{f}_1 = \overline{\partial}_n f_1 - D_k f_1$ on S (cf. [K]§23) and the conclusion follows from Theorem 3. The last part is a consequence of Corollary 2.

Let γ be the reflection introduced at the end of §2.3. The operator D_k is γ-invariant, i.e. $D_k(f \circ \gamma) = D_k(f) \circ \gamma$, since Δ is invariant. It follows a criterium for regularity of homogeneous polynomials.

Corollary 6. *Let $f = f_1 + f_2 j$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree k. Then its restriction to S extends as a regular function into B if, and only if,*

$$(N - D_k) f_1 + \overline{T(f_2)} = 0 \quad \text{on } S.$$

The same conclusion holds if there exists a pair of complex numbers $(a,b) \neq (0,0)$ such that

$$(N - D_k)(a f_1 + b f_2) + \overline{T(\bar{a} f_2 - \bar{b} f_1)} = 0 \quad \text{on } S.$$

5.2. Now we give a formulation of Corollary 4 that holds for homogeneous polynomials. Let $\mathcal{P}_k$ denote the space of homogeneous complex-valued polynomials of degree k on $\mathbb{C}^2$, and $\mathcal{H}_k$ the space of harmonic polynomials in $\mathcal{P}_k$. The space $\mathcal{H}_k$ is the sum of the pairwise $L^2(S)$-orthogonal spaces $\mathcal{H}_{p,q}$ $(p+q=k)$, whose elements are the harmonic homogeneous polynomials of degree p in z_1, z_2 and q in $\bar{z}_1, \bar{z}_2$ (cf. for example [R]§12.2). The spaces $\mathcal{H}_k$ and $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*). These spaces will be denoted by $\mathcal{H}_k(S)$ and $\mathcal{H}_{p,q}(S)$ respectively.

Let U_k^ψ be the right $\mathbb{H}$-module of (left)ψ-regular homogeneous polynomials of degree k.

Theorem 7. *For every $f_1 \in \mathcal{P}_k$, there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_2 j$ on S extends as a ψ-regular polynomial of degree at most k on $\mathbb{H}$. If $f_1 \in \mathcal{H}_k$, then $f_2 \in \mathcal{H}_k$ and $f = f_1 + f_2 j \in U_k^\psi$.*

Proof. We can suppose that f_1 has degree p in z and q in $\bar{z}$, $p+q=k$, and then extend by linearity. Let $\tilde{f}_1 = \sum_{s\geq 0} g_{p-s,q-s}$ be the harmonic extension of f_1 into B, where $g_{p-s,q-s} \in \mathcal{H}_{p-s,q-s}$ is given by formula (6). Then $\overline{\partial}_n \overline{L(g_{p-s,q-s})} = (p-s+1)\overline{L(g_{p-s,q-s})}$. We set

$$\tilde{f}_2 = \sum_{s\geq 0} \frac{1}{p-s+1} \overline{L(g_{p-s,q-s})} \in \bigoplus_{s\geq 0} \mathcal{H}_{k-2s}.$$

Then $\overline{\partial}_n \tilde{f}_2 = \overline{L(f_1)}$ on S and it follows from Theorem 3 that $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$ is a ψ-regular polynomial of degree at most k. Now it suffices to define

$$f_2 = \sum_{s\geq 0} \frac{|z|^{2s}}{p-s+1} \overline{L(g_{p-s,q-s})} \in \mathcal{P}_k$$

to get a homogeneous polynomial $f = f_1 + f_2 j$, of degree k, that has the same restriction to S as $\tilde{f}$. If $f_1 \in \mathcal{H}_k$, then $\tilde{f}_1 = f_1$, $\tilde{f}_2 = f_2$ and therefore $f \in U_k^\psi$.

Remark. The function $\tilde{f}$ in the preceding proof is the image $R(f_{1|S})$ under the action of the operator R described in §4.2. In particular, $R : \mathcal{H}_k(S) \to U_k^\psi$ is a right inverse of the restriction operator C.

Corollary 7. *(i) The restriction operator C defined on U_k^ψ induces isomorphisms of real vector spaces*

$$\frac{U_k^\psi}{\mathcal{H}_{k.0}j} \simeq \mathcal{H}_k(S), \qquad \frac{U_k^\psi}{\mathcal{H}_{k.0} + \mathcal{H}_{k.0}j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

(ii) U_k^ψ has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

Proof. The first part follows from $\ker C = \{f = f_1 + f_2 j \in U_k^\psi \ : \ f_1 = 0 \text{ on } S\} = \mathcal{H}_{k,0}j$. Part (ii) can be obtained from any of the above isomorphisms, since $\mathcal{H}_{k,0}$ (as every space $\mathcal{H}_{p,q}, p+q=k$) and $\mathcal{H}_k(S)$ have real dimensions respectively $2(k+1)$ and $2(k+1)^2$.

As an application of Corollary 7, we have another proof of the known result (see [S] Theorem 7) that the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k has dimension $\frac{1}{2}(k+1)(k+2)$ over $\mathbb{H}$.

The operator $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \to U_k^\psi$ can also be used to obtain $\mathbb{H}$-bases for U_k^ψ starting from bases of the complex spaces $\mathcal{H}_{p,q}(S)$. On $\mathcal{H}_{p,q}(S)$, R acts in the following way:

$$R(h) = h + M(h)j, \quad \text{where } M(h) = \frac{1}{p+1}\overline{L(h)} \in \mathcal{H}_{q-1,p+1} \quad (h \in \mathcal{H}_{p,q})$$

Note that $M \equiv 0$ on $\mathcal{H}_{k,0}(S)$. If $q > 0$, $M^2 = -Id$ on $\mathcal{H}_{p,q}(S)$, since $qh = \overline{\partial}_n h = -\overline{L(M(h))}$ on S, and therefore

$$h = -\frac{1}{q}\overline{L(M(h))} = -\frac{1}{q(p+1)}\overline{L}L(h) = -M^2(h).$$

If $k=2m+1$ is odd, then M is a complex conjugate isomorphism of $\mathcal{H}_{m,m+1}(S)$. Then M induces a quaternionic structure on this space, which has real dimension $4(m+1)$. We can find complex bases of $\mathcal{H}_{m,m+1}(S)$ of the form

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}.$$

Theorem 8. *Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ $(p+q=k)$. Then:*
(i) if $k=2m$ is even, a basis of U_k^ψ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q \le p \le k\}.$$

(ii) if $k=2m+1$ is odd, a basis of U_k^ψ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q=k, 0 \le q < p \le k\} \cup \{R(h_1), \dots, R(h_{m+1})\},$$

where $h_1, \dots, h_{m+1}$ are chosen such that the set

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

If the bases $\mathcal{B}_{p,q}$ are orthogonal in $L^2(S)$ and $h_1, \dots, h_{m+1} \in \mathcal{H}_{m,m+1}(S)$ are mutually orthogonal, then $\mathcal{B}_k$ is orthogonal, with norms

$$\|R(h)\|_{L^2(S,\mathbb{H})} = \left(\frac{p+q+1}{p+1}\right)^{1/2} \|h\|_{L^2(S)} \quad (h \in \mathcal{B}_{p,q})$$

w.r.t. the scalar product of $L^2(S,\mathbb{H})$.

Proof. From dimension count, it suffices to prove that the sets $\mathcal{B}_k$ are linearly independent. When $q \le p$, $q' \le p'$, $p+q = p'+q' = k$, the spaces $\mathcal{H}_{p,q}$ and $\mathcal{H}_{q'-1,p'+1}$ are distinct. Since $R(h) = h + M(h)j \in \mathcal{H}_{p,q} \oplus \mathcal{H}_{q-1,p+1}j$, this implies the independence over $\mathbb{H}$ of the images $\{R(h) : h \in \mathcal{B}_{p,q}\}$. It remains to consider the case when $k=2m+1$ is odd. If $h \in \mathcal{H}_{m,m+1}(S)$, the complex components h and $M(h)$ of $R(h)$ belongs to the same space. The independence of $\{R(h_1), \dots, R(h_{m+1})\}$ over $\mathbb{H}$ follows from the particular form of the complex basis chosen in $\mathcal{H}_{m,m+1}(S)$.

The scalar product of $L(h)$ and $L(h')$ in $\mathcal{H}_{p,q}(S)$ is

$$(L(h), L(h')) = (h, L^*L(h')) = -(h, \overline{L}L(h')) = q(p+1)(h,h'),$$

since the adjoint L^* is equal to $-\overline{L}$ (cf. [R]§18.2.2) and $\overline{L}L = q(p+1)M^2 = -q(p+1)Id$. Therefore, if h, h' are orthogonal, $M(h)$ and $M(h')$ are orthogonal in $\mathcal{H}_{q-1,p+1}$ and then also $R(h)$ and $R(h')$. Finally, the norm of $R(h)$, $h \in \mathcal{H}_{p,q}(S)$, is

$$\|R(h)\|^2 = \|h\|^2 + \|M(h)\|^2 = \|h\|^2 + \frac{1}{(p+1)^2}\|L(h)\|^2 = \frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof.

Remark. From Theorem 8 it is immediate to obtain also bases of the right $\mathbb{H}$-module U_k of left-regular homogeneous polynomials of degree k.

Examples. (i) The case $k = 2$. Starting from the orthogonal bases $\mathcal{B}_{2,0} = \{z_1^2, z_1z_2, z_2^2\}$ of $\mathcal{H}_{2,0}$ and $\mathcal{B}_{1,1} = \{z_1\bar{z}_2, z_2\bar{z}_1, |z_1|^2 - |z_2|^2\}$ of $\mathcal{H}_{1,1}$ we get the orthogonal basis

$$\mathcal{B}_2 = \{z_1^2, z_1z_2, z_2^2, z_1\bar{z}_2 - \frac{1}{2}\bar{z}_1^2 j, z_2\bar{z}_1 + \frac{1}{2}\bar{z}_2^2 j, |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j\}$$

of the six-dimensional right $\mathbb{H}$-module U_2^ψ.

(ii) The case $k = 3$. From the orthogonal bases

$$\mathcal{B}_{3,0} = \{z_1^3, z_1^2z_2, z_1z_2^2, z_2^3\}, \quad \mathcal{B}_{2,1} = \{z_1^2\bar{z}_2, z_2^2\bar{z}_1, 2z_2|z_1|^2 - z_2|z_2|^2, 2z_1|z_2|^2 - z_1|z_1|^2\},$$

$$\mathcal{B}_{1,2} = \{h_1 = z_2\bar{z}_1^2, M(h_1) = z_1\bar{z}_2^2, h_2 = 2\bar{z}_2|z_1|^2 - \bar{z}_2|z_2|^2, M(h_2) = 2\bar{z}_1|z_2|^2 - \bar{z}_1|z_1|^2\},$$

we get the orthogonal basis

$$\mathcal{B}_3 = \{z_1^3, z_1^2z_2, z_1z_2^2, z_2^3, z_1^2\bar{z}_2 - \frac{1}{3}\bar{z}_1^3 j, z_2^2\bar{z}_1 + \frac{1}{3}\bar{z}_2^3 j, 2z_2|z_1|^2 - z_2|z_2|^2 + \bar{z}_1\bar{z}_2^2 j,$$
$$2z_1|z_2|^2 - z_1|z_1|^2 - \bar{z}_1^2\bar{z}_2 j, z_2\bar{z}_1^2 + z_1\bar{z}_2^2 j, 2\bar{z}_2|z_1|^2 - \bar{z}_2|z_2|^2 + (2\bar{z}_1|z_2|^2 - \bar{z}_1|z_1|^2)j\}.$$

of the ten-dimensional right $\mathbb{H}$-module U_3^ψ.

In general, for any k, an orthogonal basis of $\mathcal{H}_{p,q}$ $(p + q = k)$ is given by the polynomials $\{P_{q,l}^k\}_{l=0,\dots,k}$ defined by formula (6.14) in [S]. The basis of U_k obtained from these bases by means of Theorem 8 and applying the reflection γ is essentially the same given in Proposition 8 of [S].

Another spanning set of the space $\mathcal{H}_{p,q}$ is given by the functions

$$g_\alpha^{p,q}(z_1, z_2) = (z_1 + \alpha z_2)^p(\bar{z}_2 - \alpha\bar{z}_1)^q \quad (\alpha \in \mathbb{C})$$

(cf. [R]§12.5.1). Since $M(g_\alpha^{p,q}) = \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1}$ for $\alpha \neq 0$ and $M(g_0^{p,q}) = -\frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1}$, where we set $g_\alpha^{p,q} \equiv 0$ if $p < 0$, from Theorem 8 we get that U_k^ψ is spanned over $\mathbb{H}$ by the polynomials

$$R(g_\alpha^{p,q}) = \begin{cases} g_\alpha^{p,q} + \frac{(-1)^q q\bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1} j & \text{for } \alpha \neq 0 \\ z_1^p\bar{z}_2^q - \frac{q}{p+1} z_2^{q-1}\bar{z}_1^{p+1} j & \text{for } \alpha = 0 \end{cases} \qquad (\alpha \in \mathbb{C},\ p + q = k)$$

Any choice of $k + 1$ distinct numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ gives rise to a basis of U_k^ψ.

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